Grade 7/8 Math Circles
February 13/14/15/16, 2023
Prime Time

What is a prime?

Division is one of the four basic mathematical operations that we deal with everyday. Today we will be looking at how we can further classify the numbers that we use and see when dividing!

Recall that a **divisor** is the number that is dividing our **dividend**, and the **quotient** is the result. For example, in the calculation $18 \div 3 = 6$, 18 is our dividend, 3 is our divisor and 6 is our quotient.

A **factor** is a divisor that gives a quotient with a remainder of 0. In the above case, both 3 and 6 are factors of 18 because $18 \div 3$ results in a whole number (6) and $18 \div 6$ results in a whole number (3). Other factors of 18 are 1, 2, 3, 6, 9 and 18.

Conversely, 18 is a **multiple** of 1, 2, 3, 6, 9, and 18 because:

$$1 \times 18 = 18,\ 2 \times 9 = 18 \text{ and } 3 \times 6 = 18.$$

Now that we have defined the terms we will be using frequently, we can classify numbers with special properties. A **prime** is a positive whole number that has exactly two factors: 1 and itself. Any other number that is *not* prime is called a **composite number**. A special case to consider is the number **1**, which is neither prime nor composite since it has exactly one factor.

Two numbers x and y are said to be **coprime** if they do not share any factors except for 1.

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Stop and Think

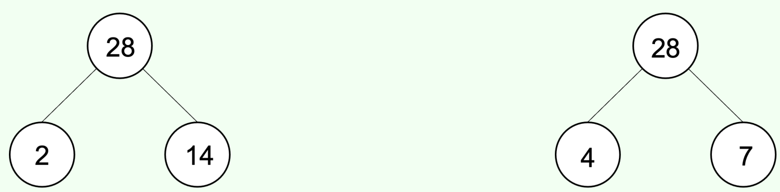
How many prime numbers are there between 1 and 30?

Prime Factorization

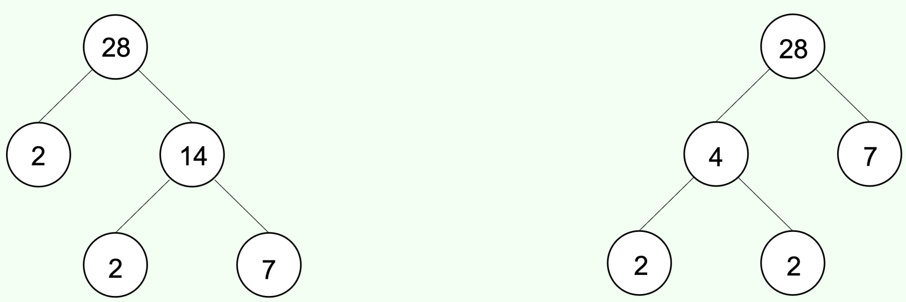
Prime numbers are useful in different areas of math, but they are most often seen in the **prime factorization** of a number. One of the most common ways of representing prime factorization is using a **factor tree**, which is a diagram showing the factors of a number. Its "leaves" are the factors, with the "branches" connecting the leaves with their respective multiples.

Example A

Most composite numbers can have multiple factor trees:



If each leaf of the tree is a composite number, we can create more branches until we are only left with leaves that are prime numbers:



This is an example of a **recursive algorithm**: a process of mathematical steps that you repeat until you reach a stopping point.

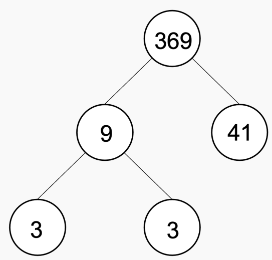
Notice how in the above example, two different prime factorization trees were created. How are they different? How are they the same?

From our prime factorization trees, we can write the **expanded form** of a number by multiplying all of the prime factors together. In the above example, we would write out $28 = 2 \times 2 \times 7$. We could also write it as $28 = 2 \times 7 \times 2$, although the convention is to write the prime factors left to right from smallest to largest, with any duplicates right beside each other.

Exercise 1

What is the prime factorization of 369? Use a factor tree in your solution.

Exercise 1 Solution



$\therefore$ the prime factorization of 369 is $3 \times 3 \times 41$.

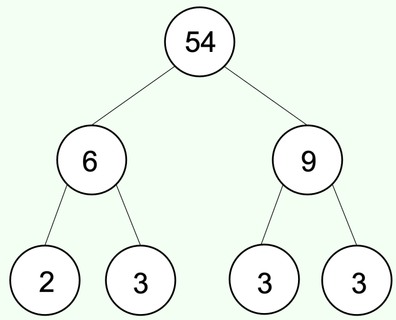
New Symbol Alert! The $\therefore$ symbol is shorthand for "therefore". It is often used in mathematical statements.

The **Unique Factorization Theorem** tells us that any number n can be expressed as a product of powers of its prime factors in exactly one way.

What do we mean by "product of powers"? It is powers being multiplied together! Recall that a power is a base raised to an exponent, of the form b^a where a is the exponent and b is the base. A few examples of powers are $2^3, 5^2, 4^1$. **Powers of a base** are the set of powers with the same base but different exponents. For instance, $3^2, 3^3, 3^4, 3^5, ...$ are all *powers of 3*.

Example B

Let's begin by creating a prime factorization tree for 54 and then seeing how we can write it as a product of powers instead:



Notice that the prime factor of 3 appears three times at the bottom of our tree. So we can rewrite $3 \times 3 \times 3$ as 3^3. The prime factor of 2 appears once at the bottom of our tree, so we do not have to rewrite it, since $2^1 = 2$. Putting everything together, we have $54 = 2 \times 3^3$.

Exercise 2

Without creating a factorization tree, write the factorization of 56 as a product of powers of its prime factors.

Exercise 2 Solution

$$\begin{aligned} 56 &= 8 \times 7 \\ &= 2 \times 4 \times 7 \\ &= 2 \times 2 \times 2 \times 7 \\ &= 2^3 \times 7 \end{aligned}$$

In mathematics we often like to *generalize* the topics that we learn. This means that we express our ideas in terms of unknown variables and values so that we can apply them to any number. For example, above we dealt with one-digit, two-digit and three-digit numbers. What about 4-digit numbers? Or 10-digit numbers? We want a way to look at prime-factorization before knowing any information. Below is a general prime-factorization for a number n:

$$n = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_k^{a_k}$$
where $a_1, a_2, \cdots, a_k$ represent the exponents corresponding to each unique prime $p_1, \cdots, p_k$

A really cool application of the prime factorization formula is that we can calculate *exactly* how many factors a certain number has!

If the prime factorization of a number is $p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_k^{a_k}$
then the number of positive factors it has is $(a_1 + 1) \times (a_2 + 1) \times \cdots \times (a_k + 1)$

Example C

The prime factorization of 24 is $2^3 \times 3$, so the number of positive factors it has is $(3 + 1) \times (1 + 1) = 8$.

We can verify this by listing out the positive factors of 24: 1, 2, 3, 4, 6, 8, 12, 24

Divisibility

So much can be done with prime factorization, but it can take a while to try and test out ALL the prime numbers. Fortunately, we have **divisibility rules** that can help us out! These rules tell us the conditions that have to be met in order for a certain number n to be divisible by other numbers.

Stop and Think

How many divisibility rules do you know?

Divisibility by	**Conditions for some number n**	**Example**
2	The ones digit is an even number (0, 2, 4, 6, 8)	$26\underline{4}$
3	The sum of the digits of n is divisible by 3	9153; $9+1+5+3=18$
4	The last two digits form a number that is divisible by 4	$8\underline{12}$
5	The ones digit is 0 or 5	$106\underline{5}$
6	n is divisible by 2 and by 3	$72 \div 2 = 36$, $72 \div 3 = 24$
7	Subtract twice the ones digit from the remaining digits of n* until you get a number that you recognize is divisible by 7	91; $9-(2\times 1)=7$
8	The last three digits are divisible by 8	$1\underline{032}$
9	The sum of the digits of n is divisible by 9	1395; $1+3+9+5=18$
10	The ones digit is 0	$345\underline{0}$
11	The alternating sum** of the digits is divisible by 11	9086; $9-0+8-6=11$
12	n is divisible by 3 and by 4	$648 \div 3 = 216$, $648 \div 4 = 162$
13	The sum of four times the unit digit and the number formed by the rest of the digits is divisible by 13	286; $28+(4\times 6)=52$

* The ones digit = the rightmost digit of a whole number. So the ones digit in 4658 would be 8, in 567 would be 7, and so on. The *remaining digits of n* is the number that is left if you "cut off" the ones digit. So the remaining digits in 4658 would be 465, and in 567 it would be 56.

** An alternating sum of numbers is one where you "take turns" adding and subtracting them. There is an invisible + in front of the first number, so we subtract the second from the first, then add the third, and so on. For example, for the number 783 we would do: $7-8+3=2$; since 2 is not divisible by 11, 783 is *not* divisible by 11.

You may wonder how all of these rules came to be, which is an excellent question for a mathematician to ask! Throughout the last couple hundred years, different mathematical researchers wanted to know how to determine the divisibility of numbers, and worked hard until they had a concrete rule. The oldest divisibility rule comes from around the year 500 C.E , from the Babylonian Talmud. The Jewish people derived it so that they could determine what year it is in the Sabbatical Cycle - a seven-year agricultural cycle oberved in Judaism. Their calculations resulted in the divisibility test for 7 that we know and use today!

Exercise 3

Is 483 divisible by 3? Is it divisible by 11?

Exercise 3 Solution

Using the divisibilty rule for 3:

$4 + 8 + 3 = 15$. Since 15 is divisible by 3 ($15 \div 3 = 5R0$) 483 is divisible by 3.

New Symbol Alert!: the uppercase R in a division statement is shorthand for "remainder".

Using the divisibility rule for 11:

Our alternating sum is $4 - 8 + 3 = -1$. Since -1 is **not** divisible by 11, 483 is not divisible by 11.

Prime Problems

There are two famous "prime problems" that mathematicians have been working on for a very long time. The first is **Goldbach's conjecture** which states that every even number greater than 2 is the sum of two prime numbers.

Stop and Think

Is this conjecture true? Can you think of any even numbers where this does not work?

It is commonly accepted that Goldbach's conjecture holds true for ALL even numbers, but there is yet to be a formal proof! As little mathematicians, it is our goal to at least convince ourselves that it is possible for this to be always true.

Example D

To get an idea of how we could justify this statement, let's consider a smaller number like 8. The only prime numbers that we can consider are 2, 3 and 5. Right away, we can see that $3 + 5 = 8$, so the conjecture holds true.

What about a larger number like 844? It would take a while to write out all of the prime numbers leading up to it. Indeed, researchers have been using computers and algorithms to verify the conjecture. As of today, Goldbach's conjecture has been proven up to 4 million trillion! While we won't go into any of the complicated formulas used to find the primes that sum to a number, we can use *mathematical reasoning*.

Back in the fourth century, the "father of geometry" Euclid concluded what we know to be **Euclid's Theorem**. It states that there are *infinitely many* prime numbers. If there is an infinite amount of prime numbers, wouldn't that mean that there is an inifnite amount of sums of prime numbers?

Stop and Think

How can Euclid's Theorem help us to understand Goldbach's Conjecture?

Let's pretend for a moment that 23 is the largest known prime. Using just 23 and our other known primes, we can make sums that add up to several even numbers:

$23 + 3 = 26$
$23 + 5 = 28$
$23 + 7 = 30$
$23 + 11 = 34$
$23 + 13 = 36$
$23 + 17 = 40$
$23 + 19 = 42$

Now imagine the infinite amount of primes that we have, all added together. That would make an infinite amount of even numbers; it is yet to be proven that these sums would cover *all* even numbers, but at least a big chunk is for certain.

Are you convinced of Goldbach's Conjecture yet? Think about it some more at home. Côte d'Azur University has an online Goldbach calculator that you can use to try and test numbers for yourself!

Exercise 4

Write out sums that satisfy Goldbach's conjecture for all the even composite numbers up to and including 20. Using the same prime number twice is allowed!

Exercise 4 Solution

$4 = 2 + 2$

$6 = 3 + 3$

$8 = 3 + 5$

$10 = 3 + 7$

$12 = 5 + 7$

$14 = 7 + 7$

$16 = 5 + 11$

$18 = 7 + 11$

$20 = 3 + 17$

The second "prime problem" that we will look at is that of "Prime Pairs". A **prime pair** is a set of two numbers that are both prime, and have a difference of 2. That is, two prime numbers a and b such that $b - a = 2$. The discussion surrounding these pairs is whether or not there exists an infinite amount of them.

Exercise 5

Write out 5 prime pairs that you can think of.

Exercise 5 Solution

3, 5 ; 5, 7 ; 11, 13 ; 17, 19 ; 29, 31

This famous problem is also called *Polignac's conjecture* after Alphonse de Polignac, who was a French mathematician that first stated it in 1846. Some developments were made in 1919, 1976, 2003, and 2013 to establish facts related to the conjecture, but none verified what the statement is trying to say. To this day, mathematicians are unable to prove whether there is an infinite amount of prime pairs, or if there is a finite list that containt a "largest prime pair". Not everything in mathematics is known, but trying to figure things out is the most fun part!

When you have *time*, check out these Prime Time Pairs for a list of prime pairs that have been converted into times!

Life of Primes

Prime numbers are cool and all, but all of these equations and formulas can seem pretty inapplicable to everyday life. After all, they are just numbers...right?

Stop and Think

Where could you be using prime factorization in your own life?

Example E

Imagine that you want to invite over four friends for your birthday, and plan on baking cupcakes to give out. The only issue is that your cupcake tins have six molds each, and you want to ensure that you bake an exact amount for all five of you to share. How many tins do you need to fill so that everyone gets an equal number of cupcakes?

Since the cupcake tins are in groups of 6, it's natural to look at the multiples of 6 and see which ones have a prime factorization that includes 5:

$6 = 2 \times 3$

$12 = 2^3 \times 3$

$18 = 2 \times 3^2$

$24 = 2^3 \times 3$

$30 = 2 \times 3 \times 5$

Since 30 is the smallest multiple of 6 that has a prime factor of 5, that is how many total cupcakes we will be baking. $30 = 5 \times 6$, so we will need to fill 5 cupcake tins.

The last example for this lesson has to do with coding and **cryptography**. Cryptography is the practice of secure communication using encoding so that information is sent and received only by those who were meant to see it. Believe it or not, prime factorization has a very important role to play in this field of mathematics.

Example F

Let's take two prime numbers and multiply them together, say 5 and 11. Then we have $5 \times 11 = 55$. Now, because we did the multiplication we know that we can write 55 as a product of two prime numbers. Using the knowledge gained from this lesson, you could probably figure this out even if I only gave you the number 55.

What about 13 and 19? We have $13 \times 19 = 247$. Could you have figured out the prime factorization if I only provided you with the number 247? Possibly, but it would take a bit more time than our last example.

What if I only provided you with the number 1189, and gave you three minutes to calculate its prime factorization? Considering that $1189 = 29 \times 41$, you may not be able to "crack the code" in time.

This is precisely how cryptography uses prime factorization: the key used to encrypt information is a number that is the product of two *very large* prime numbers. Since our knowledge of existing large prime numbers is minimal, it would take a long time for a hacker to try every possible combination!

My goal is to have *primed* you to use your newfound knowledge of these beautiful numbers in mathematics, and I hope I succeeded! If not, at least you can see that even the most advanced mathematicians have yet to uncover many things that this subject has to offer.